Maxwell’s Equations; Magnetism of Matter

32.1

This chapter will cover ranges from the basic science of electric and magnetic fields to the applied science and engineering of magnetic materials.

First, we conclude our basic discussion of electric and magnetic fields, finding that most of the physics principles in the last chapters from 21-31 can be summarized in only *four* equations, known as **Maxwell’s equations**.

Maxwell's equations describe how electric charges and electric currents create electric and magnetic fields. Further, they describe how an electric field can generate a magnetic field, and vice versa.

The **first equation** allows you to calculate the electric field created by a charge. The **second** allows you to calculate the magnetic field. The other two describe how fields 'circulate' around their sources. Magnetic fields 'circulate' around electric currents and time varying electric fields, Ampère's law with Maxwell's correction, while electric fields 'circulate' around time varying magnetic fields, Faraday's law.

Magnetic materials

Second, we examine the science and engineering of magnetic materials. The careers of many scientists and engineers are focused on understanding why some materials are magnetic and others are not and on how existing magnetic materials can be improved.

These researchers wonder why Earth has a magnetic field but you do not.

They find countless applications for inexpensive magnetic materials in cars, kitchens, offices, and hospitals and magnetic materials often show up in unexpected ways.

For example, if you have **a tattoo** and undergo an MRI (magnetic resonance imaging) scan, the large magnetic field used in the scan may noticeably tug on your tattooed skin because some tattoo inks contain magnetic particles.

Our first step here is to revisit Gauss' law, but this time for magnetic fields.

32-2 **Gauss' Law for Magnetic Fields**

Iron powder that has been sprinkled onto a transparent sheet placed above a bar magnet.

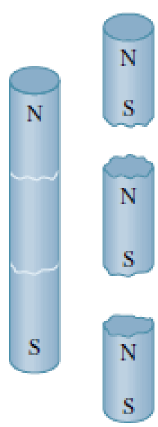
The powder grains, trying to align themselves with the magnet's magnetic field, have fallen into a pattern that reveals the field.

One end of the magnet is a *source* of the field (the field lines diverge from it) and the other end is a *sink* of the field (the field lines converge toward it).

By convention, we call the source the *north pole* of the magnet and the sink the *south pole,* and we say that the magnet, with its two poles, is an example of a **magnetic dipole.**

Suppose we break a bar magnet into pieces Fig. We should, it seems, be able to isolate a single magnetic pole, called a **magnetic monopole**.

However, we cannot - not even if we break the magnet down to its individual atoms and then to its electrons and nuclei. Each fragment has a north pole and a south pole.



The simplest magnetic structure that can exist is a magnetic dipole. Magnetic monopoles do not exist (as far as we know).

Thus: Gauss' law for magnetic fields is of saying that magnetic monopoles do not exist. This law states that the net magnetic flux ϕ_B through any closed Gaussian surface is zero:

$$\Phi_B = \oint \vec{B} \cdot d\vec{A} = 0 \qquad \text{(Gauss' law for magnetic fields).} \qquad 1$$

Contrast this with Gauss' law for electric fields,

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0} \qquad \text{(Gauss' law for electric fields).}$$

In both equations, the integral is taken over a *closed* Gaussian surface.

Gauss' law for electric fields says that this integral (the net electric flux through the surface) is proportional to the net electric charge q_{enc} enclosed by the surface. ($\phi \; \alpha \; q_{enc}$)

Gauss' law for magnetic fields says that there can be no net magnetic flux through the surface because there can be no net "magnetic charge" (individual magnetic poles) enclosed by the surface.

The simplest magnetic structure that can exist and thus be enclosed by a Gaussian surface is a dipole, which consists of both a source and a sink for the field lines.

Thus, there must always be as much magnetic flux into the surface as out of it, and the net magnetic flux must always be zero.

Gauss' law for magnetic fields holds for structures more complicated than a magnetic dipole, and it holds even if the Gaussian surface does not enclose the entire structure.

Gaussian surface II near the bar magnet of Fig. 32-4 encloses no poles, and we can easily conclude that the net magnetic flux through it is zero. Gaussian surface I is more difficult.

It may seem to enclose only the north pole of the magnet because it encloses the label N and not the label S.

However, a south pole must be associated with the lower boundary of the surface because magnetic field lines enter the surface there. (The enclosed section is like one piece of the broken bar magnet in Fig. 32-3.) Thus, Gaussian surface I encloses a magnetic dipole, and the net flux through the surface is zero.

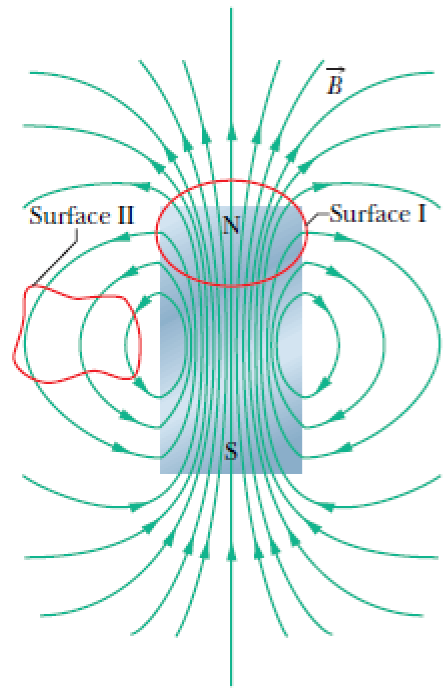


Fig. 32-4 The field lines for the magnetic field $\vec{B}$ of a short bar magnet. The red curves represent cross sections of closed, three-dimensional Gaussian surfaces.

32-3 Induced Magnetic Fields

W e know that a changing magnetic flux induces an electric field, and we ended up with Faraday's law of induction in the form

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \qquad \text{(Faraday's law of induction).} \qquad 2$$

Here E is the electric field induced along a closed loop by the changing magnetic flux encircled by that loop. Because symmetry is often so powerful in physics, we should be tempted to ask whether induction can occur in the opposite sense; that is, can a changing electric flux induce a magnetic field?

The answer is that it can; furthermore, the equation governing the induction of a magnetic field is almost symmetric with Eq. 2.

We often call it Maxwell's law of induction after James Clerk Maxwell, and we write it as

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \qquad \text{(Maxwell's law of induction).} \qquad 3$$

Here B is the magnetic field induced along a closed loop by the changing electric flux ϕ_E in the region encircled by that loop.

As an example of this sort of induction, we consider the charging of a parallel plate capacitor with circular plates. (Although we shall focus on this arrangement, a changing electric flux will always induce a magnetic field whenever it occurs.)

We assume that the charge on our capacitor (Fig. 32-5*a*) is being increased at a steady rate by a constant current i in the connecting wires. Then the electric field magnitude gbetween the plates must also be increasing at a steady rate.

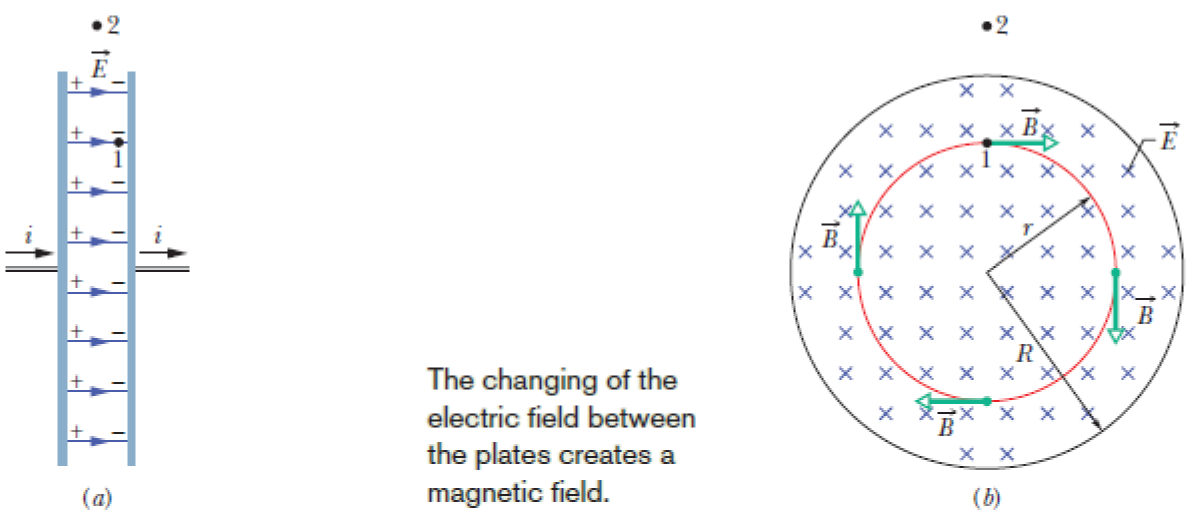


Fig. 32-5 (*a*) A circular parallel-plate capacitor, shown in side view, is being charged by a constant current *i*. (*b*) A view from within the capacitor, looking toward the plate at the right in (*a*). The electric field $\vec{E}$ is uni form, is directed into the page (toward the plate), and grows in magnitude as the charge on the capacitor increases. The magnetic field $\vec{B}$ induced by this changing electric field is shown at four points on a circle with a radius *r* less than the plate radius *R*.

Figure 32-5*b* is a view of the right-hand plate of Fig. 32-5*a* from between the plates. The electric field is directed into the page. Let us consider a circular loop through point 1 in Figs. 32-5*a* and *b*, a loop that is concentric with the capacitor plates and has a radius smaller than that of the plates. Because the electric field through the loop is changing, the electric flux through the loop must also be changing. According to Eq.32-3, this changing electric flux induces a magnetic field around the loop.

Experiment proves that a magnetic field is indeed induced around such a loop, directed as shown. This magnetic field has the same magnitude at every point around the loop and thus has circular symmetry about the central axis of the capacitor plates (the axis extending from one plate center to the other).

If we now consider a larger loop - say, through point 2 outside the plates in Figs. 32-5a and b - we find that a magnetic field is induced around that loop as well. Thus, while the electric field is changing, magnetic fields are induced between the plates, both inside and outside the gap. When the electric field stops changing, these induced magnetic fields disappear.

Although Eq. 32-3 is similar to Eq. 2, the equations differ in two ways. First, Eq. 3 has the two extra symbols $\mu_\circ$ and $\varepsilon_\circ$, but they appear only because we employ SI units. Second, Eq. 32-3 lacks the minus sign of Eq. 2, meaning that the induced electric field *E* and the induced magnetic field *B* have opposite directions when they are produced in otherwise similar situations.

To see this opposition, examine Fig. 32-6, in which an increasing magnetic field *B*, directed into the page, induces an electric field *E*. The induced field *E* is counterclockwise, opposite the induced magnetic field *B* in Fig. 32-5*b*.

Ampere–Maxwell Law

Now recall that the left side of Eq. 3, the integral of the dot product *B.ds* around a closed loop, appears in another equation - namely, Ampere's law:

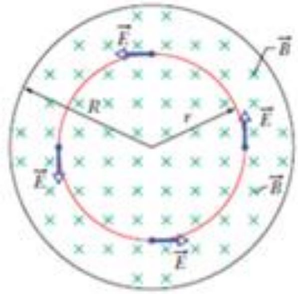
$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}} \qquad \text{(Ampere's law)}, \qquad 4$$

Where i_{enc} is the current encircled by the closed loop. Thus, our two equations that specify the magnetic field *B* produced by means other than a magnetic material (that is, by a current and by a changing electric field) give the field in exactly the same form. We can combine the two equations into the single equation

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} + \mu_0 i_{\text{enc}} \qquad \text{(Ampere–Maxwell law)}. \qquad 5$$

Fig. 32-6 A uniform magnetic field $\vec{B}$ in a circular region. The field, directed into the page, is increasing in magnitude. The electric field $\vec{E}$ induced by the changing magnetic field is shown at four points on a circle concentric with the circular region. Compare this situation with that of Fig. 32-5*b*.

The induced $\vec{E}$ direction here is opposite the induced $\vec{B}$ direction in the preceding figure.



When there is a current but no change in electric flux (such as with a wire carrying a constant current), the first term on the right side of Eq. 32-5 is zero, and so Eq. 5 reduces to Eq. 4, Ampere's law. When there is a change in electric flux but no current (such as inside or outside the gap of a charging capacitor), the second term on the right side of Eq. 5 is zero, and so Eq. 5 reduces to Eq. 3, Maxwell's law of induction.

32-4 Displacement Current

If you compare the two terms on the right side of Eq. 5, you will see that the product $\varepsilon_\circ(d\phi_E/dt)$ must have the dimension of a current. In fact, that product has been treated as being a fictitious current called the **displacement current** i_d:

$$i_d = \varepsilon_0 \frac{d\Phi_E}{dt} \quad \text{(displacement current).} \qquad 10$$

"Displacement" is poorly chosen in that nothing is being displaced, but we are stuck with the word. Nevertheless, we can now rewrite Eq. 5 as

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{d,\text{enc}} + \mu_0 i_{\text{enc}} \quad \text{(Ampere–Maxwell law),} \qquad 11$$

in which $i_{d,\text{enc}}$ is the displacement current that is encircled by the integration loop.

Let us again focus on a charging capacitor with circular plates, as in Fig. 32-7*a*. The real current i that is charging the plates changes the electric field E between the plates. The fictitious displacement current i_d between the plates is associated with that changing field E. Let us relate these two currents.

The charge q on the plates at any time is related to the magnitude E of the field between the plates at that time by Eq. 4:

$$q = \varepsilon_0 A E, \qquad 12$$

in which A is the plate area. To get the real current i, we differentiate Eq.12 with respect to time, finding

$$\frac{dq}{dt} = i = \varepsilon_0 A \frac{dE}{dt}. \qquad 13$$

To get the displacement current i_d, we can use Eq. 10. Assuming that the electric field E between the two plates is uniform (we neglect any fringing), we can replace the electric flux ϕ_E in that equation with EA. Then Eq. 10 becomes

$$i_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{d(EA)}{dt} = \varepsilon_0 A \frac{dE}{dt}. \qquad 14$$

Comparing Eqs. 13 and 14, we see that the real current i charging the capacitor and the fictitious displacement current i_d between the plates have the same magnitude:

$$i_d = i \qquad \text{(displacement current in a capacitor).} \qquad 15$$

Thus, we can consider the fictitious displacement current i_d to be simply a continuation of the real current i from one plate, across the capacitor gap, to the other plate. Because the electric field is uniformly spread over the plates, the same is true of this fictitious displacement current i_d, as suggested by the spread of current arrows in Fig. 32-7b. Although no charge actually moves across the gap between the plates, the idea of the fictitious current id can help us to quickly find the direction and magnitude of an induced magnetic field, as follows.

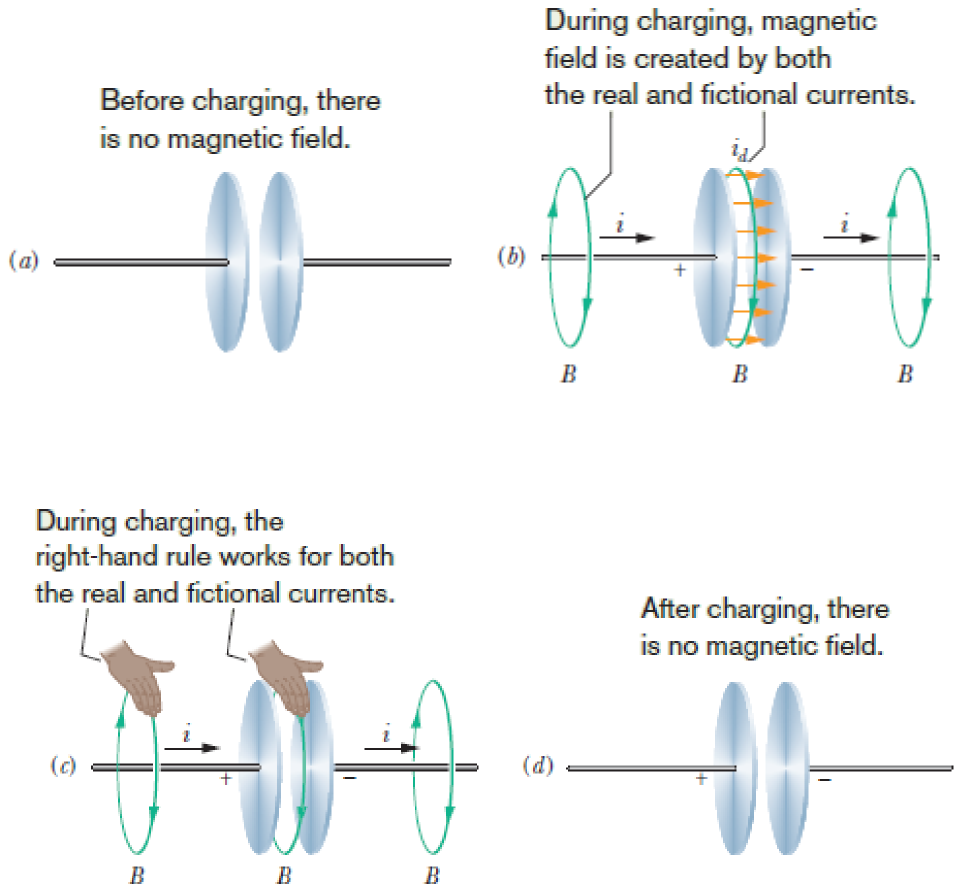


Fig. 32-7 (*a*) Before and (*d*) after the plates are charged, there is no magnetic field. (*b*) During the charging, magnetic field is created by both the real current and the (fictional) displacement current. (c) The same right-hand rule works for both currents to give the direction of the magnetic field.

Finding the Induced Magnetic Field

In Chapter 29 we found the direction of the magnetic field produced by a real current $\boldsymbol{i}$ by using the right-hand rule. We can apply the same rule to find the direction of an induced magnetic field produced by a fictitious displacement current $\boldsymbol{i_d}$, as is shown in the center of Fig. 32-7*c* for a capacitor.

We can also use $\boldsymbol{i_d}$ to find the magnitude of the magnetic field induced by a charging capacitor with parallel circular plates of radius $\boldsymbol{R}$. We simply consider the space between the plates to be an imaginary circular wire of radius $\boldsymbol{R}$ carrying the imaginary current $\boldsymbol{i_d}$. Then, from Eq. 29-20, the magnitude of the magnetic field at a point inside the capacitor at radius r from the center is

$$B = \left(\frac{\mu_0 i_d}{2\pi R^2}\right) r \qquad \text{(inside a circular capacitor).} \qquad 16$$

Similarly, from Eq. 29-17, the magnitude of the magnetic field at a point outside the capacitor at radius r is

$$B = \frac{\mu_0 i_d}{2\pi r} \qquad \text{(outside a circular capacitor).} \qquad .\ 17$$

32-5 **Maxwell's Equations**

Equation 5 is the last of the four fundamental equations of electromagnetism, called *Maxwell's equations* and displayed in Table 32-1.

These four equations explain a diverse range of phenomena, from why a compass needle points north to why a car starts when you turn the ignition key.

They are the basis for the functioning of such electromagnetic devices as electric motors, television transmitters and receivers, telephones, fax machines, radar, and microwave ovens.

Maxwell's equations are the basis from which many of the equations you have seen since Chapter 21 can be derived. They are also the basis of many of the equations you will see in Chapters 33 through 36 concerning optics.

Maxwell's Equations[a]

Name	Equation	
Gauss' law for electricity	$\oint \vec{E} \cdot d\vec{A} = q_{enc}/\varepsilon_0$	Relates net electric flux to net enclosed electric charge
Gauss' law for magnetism	$\oint \vec{B} \cdot d\vec{A} = 0$	Relates net magnetic flux to net enclosed magnetic charge
Faraday's law	$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$	Relates induced electric field to changing magnetic flux
Ampere–Maxwell law	$\oint \vec{B} \cdot d\vec{s} = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} + \mu_0 i_{enc}$	Relates induced magnetic field to changing electric flux and to current